MATH 245 S24, Exam 2 Solutions

1. Carefully define the following terms: well-ordered (for sets), recurrence.

   A set of numbers S is well-ordered if there is some ordering $<$ and S is well-ordered by $<$. A sequence is a recurrence if all but finitely many terms of that sequence are defined in terms of its previous terms.

2. Carefully state the following theorems: Nonconstructive Existence Theorem, (Vanilla) Induction Theorem.

   The Nonconstructive Existence Theorem says: To prove $\exists x \in D, P(x)$ is true, you can instead prove $\forall x \in D, \neg P(x) \equiv F$. The (Vanilla) Induction Theorem says: To prove $\forall n \in \mathbb{N}, P(n)$, you can instead prove: (a) $P(1)$; and (b) $\forall n \in \mathbb{N}, P(n) \to P(n+1)$.

3. Let $a_n = 7n^2 + 3$. Prove that $a_n = \Theta(n^2)$.

   This must be done in two parts, i.e. $a_n = O(n^2)$ and $a_n = \Omega(n^2)$.

   $a_n = \Omega(n^2)$: Choose $n_0 = 1$ and $M = 1$. Let $n \in \mathbb{N}$ with $n \geq n_0$ be arbitrary. Now $M|a_n| = 7n^2+3 \geq 7n^2 \geq n^2 = |n^2|$. hence $M|a_n| \geq |n^2|$.

   $a_n = O(n^2)$: Choose $n_0 = 2$ and $M = 8$. Let $n \in \mathbb{N}$ with $n \geq n_0$ be arbitrary. Now $n^2 \geq 2^2 = 4 \geq 3$. Hence $7n^2 + 3 \leq 7n^2 + n^2$, so $|a_n| = |7n^2 + 3| = 7n^2 + 3 \leq 8n^2 = M|n^2|$. Hence $|a_n| \leq M|n^2|$.

   ALTERNATE $a_n = O(n^2)$: Choose $n_0 = 1$ and $M = 10$. Let $n \in \mathbb{N}$ with $n \geq n_0$ be arbitrary. Now $(M-7)n^2 = 3n^2 \geq 3$. Hence $7n^2 + 3 \leq Mn^2$, so $|a_n| = |7n^2 + 3| = 7n^2 + 3 \leq Mn^2 = M|n^2|$. Hence $|a_n| \leq M|n^2|$.

4. Prove or disprove: For all $x \in \mathbb{R}$, $\lceil x \rceil \leq \lfloor x \rfloor + 1$.

   The statement is true. Let $x \in \mathbb{R}$ be arbitrary. Part of the definition of floor is $x < \lfloor x \rfloor + 1$, and part of the definition of ceiling is $\lceil x \rceil - 1 < x$. Combining and adding 1 to both sides gives $\lceil x \rceil < \lfloor x \rfloor + 2$. We now apply Thm 1.12 (since both sides of the inequality are integers), to get $\lceil x \rceil \leq (\lfloor x \rfloor + 2) - 1 = \lfloor x \rfloor + 1$.

5. Prove or disprove: For all $n \in \mathbb{Z}$, the number $\frac{n(n-1)(n-2)(n-4)}{5}$ is an integer.

   The statement is false, so we need to find a counterexample. There are plenty to choose from: 3, 8, 13, -2, -7, etc. However, we need to pick one, so let's pick $n = 3$. We calculate $\frac{n(n-1)(n-2)(n-4)}{5} = \frac{3(3-1)(3-2)(3-4)}{5} = \frac{3 \cdot 2 \cdot 1 \cdot (-1)}{5} = -\frac{6}{5}$, which is not an integer.

6. Suppose that an algorithm has runtime specified by recurrence relation $T_n = 3T_{n/2} + n^2$. Determine what, if anything, the Master Theorem tells us.

   We have $a = 3, b = 2, c_n = n^2$, and $k = 2$ because $c_n = n^2 = \Theta(n^2)$. The Master Theorem applies, since $a \in \mathbb{N}$, and b is a constant greater than 1 (need to verify/state this). We now calculate $d = \log_b a = \log_2 3$. We know that $1 = \log_2 2 < \log_2 3 < \log_2 4 = 2$, so $1 < d < 2 = k$. Hence this is the "large c_n" case, and the Master Theorem tells us that $T_n = \Theta(n^k) = \Theta(n^2)$.

7. Let $n \in \mathbb{Z}$. Prove that the following are equivalent: (a) n is even; (b) $7n$ is even; (c) $n + 1$ is odd.

   This requires at least three parts, which must be clearly labeled. Here is one way:

   (a)$\to$(b): Suppose that n is even. Then there is some $m \in \mathbb{Z}$ with $n = 2m$. We have $7n = 7(2m) = 2(7m)$. Since $7m \in \mathbb{Z}$, $7n$ is even.

   (c)$\to$(a): Suppose that $n + 1$ is odd. Then there is some $m \in \mathbb{Z}$ with $n + 1 = 2m + 1$. Subtracting 1 we get $n = 2m$, and $m \in \mathbb{Z}$, so n is even.

   (b)$\to$(c): Suppose that $7n$ is even. Then there is some $m \in \mathbb{Z}$ with $7n = 2m$. Now $7|2m$, 7 is prime, and $7 \nmid 2$. Hence $7|m$. So there is some $k \in \mathbb{Z}$ with $m = 7k$. Plugging in, we get

$7n = 2m = 2(7k) = 7(2k)$. Dividing by 2 we get $n = 2k$. Adding 1 we get $n + 1 = 2k + 1$. Since $k \in \mathbb{Z}$, $n + 1$ is odd.

ALTERNATE (b)$\rightarrow$(c), from a student solution: Suppose that $7n$ is even. Then there is some $m \in \mathbb{Z}$ with $7n = 2m$. Now $2|7n$, 2 is prime, and $2 \nmid 7$. Hence $2|n$. So there is some $k \in \mathbb{Z}$ with $n = 2k$. Adding 1 we get $n + 1 = 2k + 1$, so $n + 1$ is odd.

ALTERNATE (b)$\rightarrow$(c), from a student solution: Suppose that $7n$ is even. Then there is some $m \in \mathbb{Z}$ with $7n = 2m$. Add $-6n + 1$ to both sides, getting $n + 1 = 2m - 6n + 1 = 2(m - 3n) + 1$. Since $m - 3n \in \mathbb{Z}$, $n + 1$ is odd.

For problems 8-10, we fix unknown positive real numbers r, s, t, u, and consider the recurrence given by $x_1 = r, x_2 = s$, and $x_n = tx_{n-1} + ux_{n-2}$ (for $n \geq 3$).

8. Prove that if $\{a_n\}$ satisfies the recurrence then $a_n \leq (2M)^n$ (for all $n \in \mathbb{N}$), where $M = \max(r, s, t, u, 1)$.

We use strong induction. Two base cases: $a_1 = r \leq M \leq 2M = (2M)^1$, and $a_2 = s \leq M \leq 2M \leq (2M)^2$ (since $2M \geq 1$). Now let $n \in \mathbb{N}$ with $n \geq 3$, and assume that $a_{n-1} \leq (2M)^{n-1}$ and $a_{n-2} \leq (2M)^{n-2}$. We now have $a_n = ta_{n-1} + ua_{n-2} \leq t(2M)^{n-1} + u(2M)^{n-2} \leq t(2M)^{n-1} + u(2M)^{n-1} \leq M(2M)^{n-1} + M(2M)^{n-1} = (2M)(2M)^{n-1} = (2M)^n$. Hence $a_n \leq (2M)^n$.

9. Prove that there exists at least one sequence $\{a_n\}$ satisfying this recurrence (with indices in $\mathbb{N}$, i.e. $n \in \mathbb{N}$). NOTE: Just prove it exists, do not try to find a closed form.

This problem seems to have filled many of you with existential dread. It's proving one of the basic facts that we use in chapter 7.1, that recurrences (of this type) must have solutions. Not every recurrence does! For example, consider: $x_1 = -3, x_n = \frac{1}{3+x_{n-1}} (n \geq 2)$. This one grinds to a halt immediately, there is no way to even get a second term much less infinitely many afterward.

Using the techniques of Chapter 7.1 is not the pathway to success here. At best, you would get a candidate sequence $a_n = Ar_1^n + Br_2^n$ or perhaps $a_n = Ar_1^n + Bnr_2^n$. But that's only a candidate, you have not proved that it satisfies the recurrence! You would need to prove $a_1 = r$, and $a_2 = s$, and $a_2 = ta_{n-1} + ua_{n-2}$ (for all $n \geq 3$), which is awkward to do since you don't know A, B, r_1, or r_2 explicitly. That is why the problem told you not to try to find a closed form.

SOLUTION: We prove $\{a_n\}$ exists by strong induction. Two base cases, take $a_1 = r$ and $a_2 = s$, which satisfies the recurrence's initial conditions. Now, let $n \in \mathbb{N}$ with $n \geq 3$. We assume that a_{n-1} and a_{n-2} have been chosen already, so they must exist (and satisfy the recurrence). We now choose $a_n = ta_{n-1} + ua_{n-2}$, which exists and also satisfies the recurrence.

10. Prove that there exists at most one sequence $\{a_n\}$ satisfying this recurrence (with indices in $\mathbb{N}$, i.e. $n \in \mathbb{N}$). HINT: minimum element induction on a set of indices.

Suppose $\{a_n\}$ and $\{a'_n\}$ are two sequences, that each satisfy the recurrence. We now define set $S = \{m : a_m \neq a'_m\}$, the set of indices where the two sequences are not equal. Note that S has a lower bound of 3, since $a_1 = a'_1 = r$ and $a_2 = a'_2 = s$. If S is empty, then we are done and happy (the two sequences are equal). If instead S is not empty, then by minimal element induction S has a minimal element $n \geq 3$. Hence $a_n \neq a'_n$, but $a_{n-1} = a'_{n-1}$ and $a_{n-2} = a'_{n-2}$. But now $a_n = ta_{n-1} + ua_{n-2} = ta'_{n-1} + ua'_{n-2} = a'_n$. This is impossible, we cannot have both $a_n \neq a'_n$ and $a_n = a'_n$. Hence S must be empty, and $\{a_n\} = \{a'_n\}$.